Discrete symmetries of monopole bundles

1992 J. Phys. A: Math. Gen. 25 899

(http://iopscience.iop.org/0305-4470/25/4/026)

J. Phys. A: Math. Gen. 25 (1992) 899-908. Printed in the UK

Discrete symmetries of monopole bundles

A B Ryzhov and A G Savinkov

P N Lebedev Physical Institute, Leninsky propsect, 53, Moscow B-333, 117924, USSR

Received 13 March 1991, in final form 13 August 1991

Abstract. We point out that the very notion of space reflection is ill-defined when physical states are defined on a fibre bundle describing a charge-Dirac monopole system. In other words, there is no lift of the space reflection operator to the total space of such a non-trivial bundle.

We construct a well defined transposition operator of two dyons on the non-trivial two-dyon bundle; consequently, we can correctly define its action on local sections. It is shown that symmetric wavefunctions defined on this bundle cannot be transformed into antisymmetric ones by a gauge transformation, in contradiction to the well known statement first pointed out in connection with the dyon spin problem.

1. Introduction

In the context of a trivial bundle, commonly used in physics, whose total space is equivalent to $B\times F$, the Cartesian product of the configuration space B with some space F connected with some gauge degrees of freedom, there are no difficulties in formulating how a space-symmetry operator acts on sections (wavefunctions). Thus, states of a system defined on such a bundle correspond to global sections $\psi(X)$ (i.e. $\psi: B\to B\times F$ for which $p\psi$ is an identity on the base B, where p is the projection $B\times F\to B$) which may be subjected to the same operations as ordinary wavefunctions.

But in the context of a system with a non-trivial bundle, i.e. one which can be represented by a Cartesian product $U_\alpha\times F$ only over shrinkable regions U_α of the base B (but not over the whole of B), the states are not ordinary functions (see e.g. [1, 2]). They are local sections. In our case when the gauge group is U(1) they are local sections of a complex line bundle which we denote by LD. In an equivalent description [3], physical states can be represented as ordinary functions Ψ defined on the total space of the principal bundle D (with U(1) as fibre) associated with LD, and satisfying the equivariant condition [4]

$$\Psi(Zg)=g^{-1}\Psi(Z) \qquad Z\in D \qquad g\in \mathrm{U}(1). \tag{1}$$

So, to define a simple operator on the bundle, say the space parity operator, we must 'lift' the operator $\zeta: X\to -X$ defined on the base to an operator $\hat{\zeta}$ defined on the total space D. It is evident, at least mathematically, that $\hat{\zeta}$ must be an automorphism of the bundle, i.e. a mapping $\hat{\zeta}: D\to D$ preserving the bundle structure (the group U(1) action),

$$\hat{\zeta}(Zg)=(\hat{\zeta}(Z))g \qquad \forall Z\in D. \tag{2}$$

The non-existence of this lift $\hat{\zeta}$ (which will be proved in section 2) leads to the absence of the correct notion of 'space parity' for the states defined on this bundle, and though it exists for the states globally defined on the base.

0305-4470/92/040899+10$04.50

Thus, in the case of non-trivial bundles, one must always solve the problem of lifting. This same problem occurs in the construction of the dyon transposition operator for the two-dyon system. We will show that symmetric wavefunctions cannot be transformed into antisymmetric ones by a gauge transformation, in contradiction to the well known statement of Goldhaber [5] on the dyon statistics problem.

2. Non-existence of a linear space reflection operator on the bundle attributed to a Dirac monopole

Since we are working with monopole bundles, it is worth recalling some essential features. The system of a Dirac monopole and a charge has two local Hamiltonians (see e.g. [1]) due to the two choices of the monopole potential

$$A_+\left(A_{+r}=0,\ A_{+\theta}=0,\ A_{+\varphi}=\frac{\mu}{2}\tan\frac{\theta}{2}\right)$$

and

$$A_-\left(A_{-r}=0,\ A_{-\theta}=0,\ A_{-\varphi}=-\frac{\mu}{2}\cot\frac{\theta}{2}\right)$$

corresponding to the domains $U_-=\mathbb{R}^3\backslash\{x=y=0,\ z\in[0,+\infty)\}$ and $U_+=\mathbb{R}^3\backslash\{x=y=0,\ z\in(-\infty,0]\}$. The transition from the wavefunction ψ_+ on U_+ to the wavefunction ψ_- on U_- is determined by the transition function

$$T_{+-}=\exp(2\mathrm{i}e\mu\varphi(\boldsymbol{r}))$$

where $\varphi(\boldsymbol{r})$ is the azimuthal angle of the vector $\boldsymbol{r}$ from the monopole to the charge. Thus, a state of the system is described by a *pair* of functions (each has its own domain) connected on the overlap $U_+\cap U_-$ by the transition function T_{+-} given by

$$\psi_+=T_{+-}\psi_-=\exp(2\mathrm{i}e\mu\varphi(\boldsymbol{r}))\psi_-.$$

This means that we have a complex-line fibre bundle with the base $B=\mathbb{R}^3\backslash\{0\}$ and pairs of functions ψ_+, ψ_- connected by T_{+-}, which make up a section of this bundle, i.e. a map $\psi_\pm:\mathbb{R}^3\backslash\{0\}|_{U_\pm}\to LD$ satisfying $p\psi_\pm=1$ on the base (where p is the projection of LD to the base). Alternatively, one can work with a principal fibre bundle D associated with LD in which case the pairs ψ_+ and ψ_- are represented by complex functions Ψ on D. The ingredients of this principal bundle are [4]:

(i) the total space $D=\mathbb{C}^2\backslash\{0\}$,

(ii) action U(1) on D: $\exp(\mathrm{i}\varphi)(Z_1,Z_2)=(\exp(\mathrm{i}\varphi)Z_1,\ \exp(\mathrm{i}\varphi)Z_2)$,

(iii) projection p of D to the base B is

$$p(\boldsymbol{Z})=\boldsymbol{X},\ X_i=\bar{Z}\sigma_i Z,\ \boldsymbol{Z}=\begin{pmatrix}Z_1\\ Z_2\end{pmatrix}$$

(iv) points (i-iii) are identical for all integer $n=2e\mu$, the distinction lies in the choice of the class of functions Ψ: $D\to\mathbb{C}$, which is defined by the equivariant condition

$$\Psi(Z\exp(\mathrm{i}\varphi),\bar{Z}\exp(-\mathrm{i}\varphi))=\exp(\mathrm{i}n\varphi)\Psi(Z,\bar{Z}).$$

All transformations of states in the system with $2e\mu=n$ must leave Ψ in the same class, i.e. preserve its equivariance condition. If a transformation of Ψ is generated by a diffeomorphism f of D, then (it is a simple exercise) perserving the equivariance condition means that f is an automorphism of the bundle

$$f(Zg)=f(Z)g \qquad Z\in D \qquad g\in \mathrm{U}(1).$$

For simplicity, let us examine the fibre bundle D for fixed $r=\bar{Z}Z$. It is the well known Hopf fibring $S^3 \to S^2$. If the lift $\hat{\zeta}$ of the space reflection ζ existed, i.e. diagram (3) commuted,

$$\begin{array}{ccc} S^3 & \xrightarrow{\hat{\zeta}} & S^3 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^2 & \xrightarrow{\zeta} & S^2 \end{array} \tag{3}$$

then $\hat{\zeta}$ would be a one-to-one mapping. Hence its degree (winding number), deg $\hat{\zeta}$, would be ± 1. Let us now prove that this cannot be negative, i.e. deg $\hat{\zeta} \neq -1$. Since ζ has no fixed points, then $\hat{\zeta}$ cannot have them and consequently its Lefschetz number (the algebraic number of fixed points) is $\mathrm{Lef}(\hat{\zeta})=0$. But the map $J: S^3 \to S^3$, defined by $J(X_1, X_2, X_3, X_4) = (X_1, -X_2, -X_3, -X_4)$ with deg $J=1$, has two fixed points $(1,0,0,0)$ and $(-1,0,0,0)$; so, its Lefschetz number is $\mathrm{Lef}(J)=2$. The difference of Lefschetz numbers for J and $\hat{\zeta}$ convinces us that J and $\hat{\zeta}$ cannot be in the same homotopy class and consequently have different degrees. So, the only possibility is deg $\hat{\zeta}=1$.

The degree of ζ is (-1), so, to get deg $\hat{\zeta}=1$, the restriction of $\hat{\zeta}$ to the fibres must have degree (-1) (the fibres are the orbits of the group U(1), oriented by its action). But the last property contradicts condition (2) because (2) implies the conservation of the orientation.

Hence the only possible way to lift ζ is by violating property (2). For instance, the change of orientation can be realized by the condition

$$\hat{\zeta}(Zg) = \hat{\zeta}(Z)g^{-1} \tag{4}$$

(i.e. $\hat{\zeta}$ is an 'antiautomorphism'). The operator of this kind $\hat{\zeta}:(Z_1, Z_2) \to (-\bar{Z}_2, \bar{Z}_1)$ is well known [6]. Such $\hat{\zeta}$ transforms n-equivariant functions Ψ to $(-n)$-equivariant ones (it is similar to CP). To restore n-equivariance one can conjugate Ψ. Thus, on this bundle the lift (corresponding to condition (4)) of space reflection combined with time inversion T, which implies the conjugation of wavefunction, conserves the Schrödinger equation.

There is another way to see the non-existence of a lift $\hat{\zeta}$ of the space reflection. Assume that $\hat{\zeta}$ exists. The commutative diagram (5) would then exist:

$$\begin{array}{ccc} S^3 & \xrightarrow{\hat{\pi}} & S^3/\hat{\zeta} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ S^2 & \xrightarrow{\pi} & S^3/\zeta = \mathbb{R}P^2. \end{array} \tag{5}$$

A simple topological consideration shows, however, that every principal U(1)-bundle over $\mathbb{R}P^2$ is trivial, and thus the bundle $S^3/\hat{\zeta} \xrightarrow{p'} \mathbb{R}P^2$ carries a flat connection ω_0. Therefore, the connection $\omega = \hat{\pi}^*\omega_0$ on the bundle is also flat. This contradicts the non-triviality of the Hopf bundle $S^3 \xrightarrow{p} S^2$.

3. Transposition operator on the non-trivial two-dyon fibre bundle

For a long time the system consisting of a monopole and a charge (dyon) has been an example of how a spin is generated by two-particle interaction, in particular, of how bosons can make up a fermion. The statistics problem of dyons was examined in Goldhaber's well known work [5], referred to in many reviews and popular lectures (see e.g. [7, 8]).

The monopole-charge system has a dynamical integral of motion $\boldsymbol{J}$, whose components satisfy the commutation relations of the su(2)-algebra. It is widely asserted that through the monopole-charge interaction a spin $n/2 = e\mu$ (lowest j) state is generated. If $n/2$ were a spin of the system, we would be able to solve a puzzle connected with spin and statistics (when n is odd), i.e. to make a fermion from two bosons (a spinless particle and a spinless Dirac monopole). Goldhaber [5] provided the solution to this problem in his study of the behaviour of the two-dyon wavefunction under transpositon. The puzzle is resolved by the following result. There are two gauge-equivalent descriptions of the two-dyon system. In the first case with symmetric wavefunctions ψ and some Hamiltonian H, the dyons are regarded as bosons. In the second one with antisymmetric wavefunctions ψ' and a Hamiltonian H', the dyons are regarded (for n odd) as fermions. These descriptions are connected by a gauge transformation.

However, the approach proposed in this work of Goldhaber does not take into account that the system including Dirac monopoles does not have a global wavefunction on the configuration space. (There are some configurations on which Hamiltonians and corresponding Schrödinger equations considered in [5] are not defined—when monopoles and charges are located along the Z-axis). The behaviour of a *local function* under dyon transposition is not a reason for concluding that a quantum state is fermionic or bosonic. The very possibility of obtaining a local function of arbitrary symmetricity confirms this. Rather, the sections of a complex line bundle should be examined. Using the correct approach, we consider the two-dyon bundle and construct the operator of dyon transposition defined on this bundle. As a consequence, we conclude that the behaviour of a *section* under dyon transpostion is a proper definition of the bosonic and fermionic quantum states.

In the two-dyon case, in a natural way, one obtains 16 respective domains and Hamiltonians corresponding to the choices $\boldsymbol{A}_+$ and $\boldsymbol{A}_-$ for each of four variables (see figure 1):

$$\boldsymbol{r}_1,\ \boldsymbol{r}_{12} = \boldsymbol{r}_2 + \boldsymbol{R}_2 - \boldsymbol{R}_1 \qquad \boldsymbol{r}_2,\ \boldsymbol{r}_{21} = \boldsymbol{r}_1 + \boldsymbol{R}_1 - \boldsymbol{R}_2.$$

The transition map between wavefunctions defined on different domains (local sections) is a combination of the $\exp(2ie\mu\varphi)$; for instance,

$$\Psi_{+++-} = \Psi_{+--+}\exp(2ie\mu[\varphi(\boldsymbol{r}_{12}) + \varphi(\boldsymbol{r}_2) - \varphi(\boldsymbol{r}_{21})]).$$

Thus, the two-dyon bundle has the base

$$M2 = \{(\boldsymbol{r}_1, \boldsymbol{R}_1, \boldsymbol{r}_2, \boldsymbol{R}_2) | \boldsymbol{r}_1 \neq 0, \boldsymbol{r}_{12} \neq 0, \boldsymbol{r}_2 \neq 0, \boldsymbol{r}_{21} \neq 0\}$$

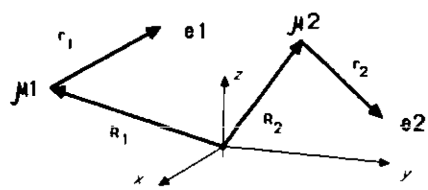

Figure 1. Monopoles $\mu 1$, $\mu 2$ with radius-vectors $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ and spinless charged particles $e1$, $e2$ with $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ being vectors from respective monopoles to them.

and this base is covered by the 16 regions

$\{U_\alpha | \alpha = (\pm\pm\pm\pm)\}$, $U_\alpha = \{(\boldsymbol{r}_1, \boldsymbol{R}_1, \boldsymbol{r}_2, \boldsymbol{R}_2) \in M2 | \boldsymbol{r}_1 \in U_\pm, \boldsymbol{r}_{12} \in U_\pm, \boldsymbol{r}_2 \in U_\pm, \boldsymbol{r}_{21} \in U_\pm\}$

with transition functions $T_{\alpha\beta}$. (In fact two of the regions, $(+-+-)$ and $(-+-+)$, are empty because of a linear dependence $\boldsymbol{r}_1 - \boldsymbol{r}_{12} = \boldsymbol{r}_{21} - \boldsymbol{r}_2$.)

Equivalently, instead of sections of the aforementioned complex line bundle (let us denote it as $LD2$), one makes use of functions on the total space of the associated principal U(1)-fibre bundle $D2$. The states of the system are described either by 16 functions, each pair connected by the transition function $T_{\alpha\beta}$ on the overlap of their domains, or by a single function defined on the total space $D2$ (and satisfying condition (9)).

The problem of determining the lift to $D2$ of an arbitrary element of the group Diff($M2$) of diffeomorphisms of the base $M2$ is connected with the exact sequence

$$1 \to \mathrm{Aut}_V(D2) \xrightarrow{i_*} \mathrm{Aut}(D2) \xrightarrow{p_*} \mathrm{Diff}(M2) \tag{6}$$

where

$$\mathrm{Aut}(D2) = \{f | f(ug) = f(u)g,\ u \in D2,\ g \in \mathrm{U}(1)\}. \tag{7}$$

$\mathrm{Aut}_V(D2)$ is the group of gauge transformations, i.e. the subgroup of the group Aut($D2$) consisting of elements f which project to the identity transformation on the base: $p_* f = 1$ (the homomorphism p_* is induced by the projection p: $D2 \to M2$).

The dyon transposition operator is an automorphism $\hat{\tau}$ of the fibre bundle, satisfying $\hat{\tau}^2 = 1$ which is the lift of the mapping

$$\tau : (\boldsymbol{r}_1, \boldsymbol{R}_1, \boldsymbol{r}_2, \boldsymbol{R}_2) \to (\boldsymbol{r}_2, \boldsymbol{R}_2, \boldsymbol{r}_1, \boldsymbol{R}_1) \tag{8}$$

defined on the base. If the lift $\hat{\tau}$ for $\tau \in \mathrm{Diff}(M2)$ existed, then $\hat{\tau}$ would belong to Aut($D2$). Since we are interested in how $\hat{\tau}$ acts on local sections ψ of the line bundle $LD2$ or, equivalently, on functions $\Psi(u)$ defined on the principal bundle $D2$ which satisfy the equivariance condition

$$\Psi(ug) = g^{-1}\Psi(u) \tag{9}$$

where $g \in \mathrm{U}(1)$, $u \in D2$ and $\psi_\alpha(X) = (\sigma_\alpha^* \Psi)(X) = \Psi(\sigma_\alpha(X))$ for local sections σ_α in $D2$, it is important to know how $\hat{\tau}$ acts over the regions U_α which cover the base. It is evident that $\tau : U_\alpha \to U_{\alpha^c}$, where $\alpha^c := (klij)$ if $\alpha = (ijkl)$. So, it is convenient to define the automorphism $\hat{\tau}$ in the form

$$(g, X)_\alpha \to (h_{\alpha^c}(X^c)g, X^c)_{\alpha^c} \qquad X^c := (\boldsymbol{r}_2, \boldsymbol{R}_2, \boldsymbol{r}_1, \boldsymbol{R}_1)$$

where $X = (\boldsymbol{r}_1, \boldsymbol{R}_1, \boldsymbol{r}_2, \boldsymbol{R}_2)$.

One can show that the correspondence

$$(g, X)_\alpha \to (g, X^c)_{\alpha^c}$$

correctly defines an automorphism $\hat{\tau} \in \mathrm{Aut}(D2)$ which satisfies $\hat{\tau}^2 = 1$ on the two-dyon fibre bundle $D2$. Then

$$\hat{\tau} : (g, X)_\alpha \to (T_{\alpha\alpha^c}(X^c)g, X^c)_\alpha \tag{10}$$

for $X \in U_\alpha \cap U_{\alpha^c}$. Therefore, on the sections

$$\hat{\tau}\psi_\alpha(X) := \psi_\alpha(X^c)T_{\alpha^c\alpha}(X^c).$$

For example, when $\alpha=(++++)$, $\alpha^c=\alpha$, $T_{\alpha^c\alpha}\equiv 1$, one has

$$\hat{\tau}\psi_{++++}(X):=\psi_{++++}(X^c)$$

but when $\alpha=(+-++)$, one obtains

$$\hat{\tau}\psi_{+-++}(X):=\exp(2ie\mu\pi[\varphi(\boldsymbol{r}_{12})-\varphi(\boldsymbol{r}_{21})])\varphi_{+-++}(X^c).$$

The symmetry condition $\hat{\tau}\Psi=\Psi$ of a global function over these regions has the form

$$\psi_\alpha(X^c)=\psi_\alpha(X) \qquad \text{for } \alpha=(++++)$$

and

$$\psi_\beta(X^c)=\exp(2ie\mu\pi[\varphi(\boldsymbol{r}_{12})-\varphi(\boldsymbol{r}_{21})])\psi_\beta(X) \qquad \text{for } \beta=(+-++).$$

When $|\boldsymbol{R}_1-\boldsymbol{R}_2|\to\infty$, the right-hand side of the last equality tends to $\exp(2ie\mu\pi)$ $\psi_\beta(X)=-\psi_\beta(X)$ for $2e\mu$ odd.

The transition from ψ_α which satisfies $\psi_\alpha(X^c)=\psi_\alpha(X)$ to $\psi_\beta(X)$ which satisfies $\psi_\beta(X^c)=-\psi_\beta(X)$ was interpreted in [5] by saying that the dyons are fermions (for $2e\mu$ odd). Now we have different forms of the single symmetry condition $\hat{\tau}\Psi=\Psi$ for different local sections. In general, on a non-trivial fibre bundle, a global section can be represented by symmetric and antisymmetric functions on the respective regions. The Möbius band is the simplest example (see figure 2).

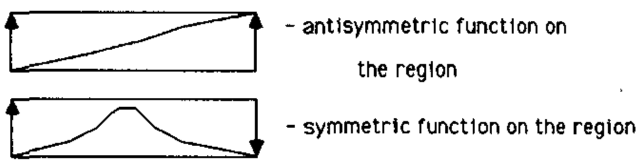

Figure 2. Antisymmetrical and symmetrical functions on two charts of a circle (the base of the Möbius band), which compose a global section.

Note that on a fibre bundle, the notion of being symmetric is properly formulated with respect to an automorphism $\hat{\tau}$. The property $\hat{\tau}\Psi=\Psi$ is the definition of a symmetric Ψ, and $\hat{\tau}\Psi=-\Psi$ is the definition of an antisymmetric one. Hence, a symmetric global function (when we pass to the language of local sections) can take an antisymmetric form on certain regions. At first sight, this fact seemingly contradicts the Pauli principle namely, that an antisymmetric function must be equal to zero at $\boldsymbol{r}_1=\boldsymbol{r}_2$, $\boldsymbol{R}_1=\boldsymbol{R}_2$ in contrast to the behaviour of a symmetric one. But on the two-dyon fibre bundle this is not a problem, since a symmetric wavefunction takes an antisymmetric local form only when $|\boldsymbol{R}_1-\boldsymbol{R}_2|\to\infty$.

Let us now return to the assertion in [5] that a symmetric wavefunction of two identical dyons can be converted by a gauge transformation into an antisymmetrical form when $n=2e\mu$ is odd. The exactness of the sequence (6) implies that if there exists some lift on $D2$ of τ defined on the base, then we have another exact sequence

$$1\to \mathrm{Aut}_V(D2)\xrightarrow{i_*} E \underset{s}{\overset{p_*}{\rightleftarrows}} \{1,\tau\}\to 1 \tag{11}$$

where E denotes the subgroup of $\mathrm{Aut}(D2)$, which is the extension of the group $T=\{1,\tau\}$ by $\mathrm{Aut}_V(\mathrm{D2})$. In addition, in order to satisfy the condition $\hat{\tau}^2=1$, the homomorphism p_* must have the right inverse s, $p_*s=1$ (i.e. the exact sequence (11) has to split). Let us prove that this condition is satisfied.

It is evident that any lift $\tilde{\tau}$ ($\tilde{\tau} = \text{lift}(\tau)$) satisfies $\tilde{\tau}^2 = \alpha \in \text{Aut}_V(D2)$, where α may be regarded as a function on the base, $\alpha = \alpha(pu) = \alpha(X)$ which follows from the fact that U(1) is Abelian. The base of the two-dyon fibre bundle $M2$ can be represented as the subset

$$M2 = \bigcup_{(\boldsymbol{R}_1, \boldsymbol{R}_2) \in \mathbb{R}^3 \times \mathbb{R}^3} \{(\boldsymbol{R}_1, \boldsymbol{R}_2) \times (\mathbb{R}^3 \backslash \{\boldsymbol{R}_1, \boldsymbol{R}_2\}) \times (\mathbb{R}^3 \backslash \{\boldsymbol{R}_1, \boldsymbol{R}_2\})\}$$

of $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. One can show that the set of homotopy classes of mappings $\alpha : (M2 \to \text{U}(1)$ is trivial. It follows that $\gamma = arg(\alpha)$ is a function $\gamma.\ M2 \to \mathbb{R}$ for arbitrary $\alpha(X)$. This fact is sufficient for the sequence (11) to split. Actually, let $\tilde{\tau}^2 = \alpha(X) = \exp(\text{i}\gamma(X))$, then using the associativity in E, one can obtain $\tilde{\tau} \cdot \exp(\text{i}\gamma(X)) = \exp(\text{i}\gamma(X)) \cdot \tilde{\tau}$ (i.e. $\gamma(X^c) = \gamma(X)$). Let us define $\hat{\tau} = \tilde{\tau} \exp(-\text{i}\gamma(X)/2)$. Then $\hat{\tau}^2 = 1$.

Having the operator $\hat{\tau}$ ($\hat{\tau}^2 = 1$), one can extract a symmetric and an antisymmetric function on the total space $D2$, $V_{\pm} = \{\Psi | \hat{\tau}\Psi = \pm\Psi\}$, which satisfy the equivariance condition (9). It is evident that there is no gauge transformation ($\alpha \in \text{Aut}_V(D)$) from any $\Psi_+ \in Y_+$ to $\Psi_- \in Y_-$ for in this case $\alpha^{-1}(X)$ would vanish on the whole set of fixed points

$$M_0 = \bigcup_{(\boldsymbol{R}, \boldsymbol{R}) \in \mathbb{R}^3 \times \mathbb{R}^3} \{(\boldsymbol{R}, \boldsymbol{R}) \times \text{diag}(\mathbb{R}^3 \backslash \{\boldsymbol{R}\}) \times (\mathbb{R}^3 \backslash \{\boldsymbol{R}\})\} \subset M2$$

of the involution τ on the base $M2$.

In the spirit of Souriau [9], we can also view the distinction between fermion and boson spaces as follows. If the particles are identical, the interchange operation ($\mathbb{Z}_2$) acts on two-dyon space $M1 = M2/M_0$. Saying that the two dyons are identical means that the 'true' configuration space should be $M1/\mathbb{Z}_2$ which is not simply connected: $\pi_1(M1/\mathbb{Z}_2) = \mathbb{Z}_2$. The action of $\mathbb{Z}_2$ on the base lifts to the U(1) 'prequantum' bundle in two different ways with the consequence that the non-simply connected $M1/\mathbb{Z}_2$ admits two distinct prequantizations corresponding to bosons and fermions.

Summarizing the above discussion, we can assert that the arguments in favour of a gauge equivalence [5] of two descriptions of the dyon as a boson and as a fermion are not correct in principle. There exist the well defined Hilbert spaces of boson and fermion states. The non-trivial structure of the fibre bundle affects the definition of the dyon transposition operator, but it does not affect the customary view. In the framework of quantum mechanics, the choice of statistics is taken from experience or it follows from relativistic quantum theory. The choice of the proper symmetricity of a wavefunction on the two-dyon fibre bundle with a given $e\mu$ may be completely defined only in relativistic quantum theory in the context of an explicit construction of multidyon fibre bundles.

Nevertheless, a quantum mechanical counter-argument to the statement 'the dyon is a fermion' follows from the Zeeman effect in a weak field for the system formed by a spinless charge and a spinless monopole [10]. The splitting for a dyon differs significantly from the splitting of levels for an ordinary fermion.

In [11], the definition of a dyon permutation is in accordance with ours, but the conclusions are different. The authors tried to reverse the statistics in another way (not by a gauge transformation). Their method is based on the independence of $\int_\gamma A(X)\,\text{d}s$ (where $A(X)$ is a connection) on the path γ in the chart of an 'asymptotic' fibre bundle. This independence is obtained by taking the surface S 'bounded by paths γ_1 and γ_2' and integrating over it. But such a surface S does not exist, because by reducing U_{++++} (it is that which was done in [10]) to the 'asymptotic' subset U_{++++} as consisting of the configurations with $\boldsymbol{r}_1$, $\boldsymbol{r}_2 \ll \boldsymbol{r}_{12}$, $\boldsymbol{r}_{21}$, the authors lose the simple connectedness of the region. It is not difficult to prove in fact (see the appendix), that the double dyon permutation (2π clockwise rotation, considered as a loop on 'asymptotic' configuration space B^{as}) is not contractible on $U_{+++}{}^{as}$ (see figure 3). It follows that the deformation of this rotation in B^{as} to the identity also includes configurations where the field $F = dA$ cannot be neglected. *That is why the dependence on the path is essential and a reversal* of statistics does not occur.

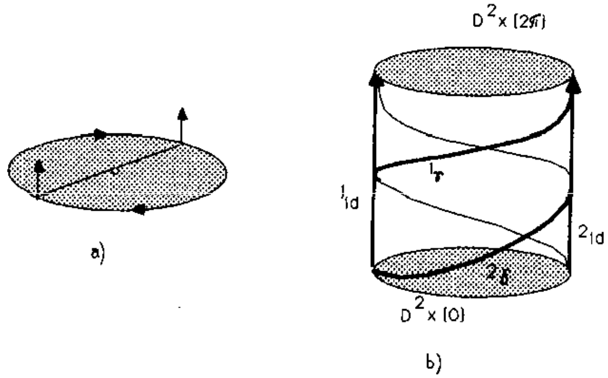

Figure 3. (*a*) 2π clockwise rotation (it is a path in configuration space). (*b*) The pair of curves $(1_\gamma, 2_\gamma)$ represents 2π rotation, the pair $(1_{id}, 2_{id})$-identical path.

In a subsequent paper [12], Friedman and Sorkin investigated an *asymptotic* spin-statistics theorem for dyons. They introduced an 'asymptotic equivalence' of bundles and examined the statistics on the non-interacting dyons bundle which is *asymptotically* equivalent to the bundle of interacting dyons. Such a transition to the bundle of non-interacting dyons differs from the standard one where the states of both interacting and free particles are defined on the same (trivial) bundle.

In the case of interaction of a monopole and a charge, the product of their charges is quantized and the space of wavefunctions corresponding to a topological charge $n \neq 0$

$$\Im_n = \{\Psi(Z, \bar{Z}) \,|\, \Psi(Z\, e^{i\alpha}, \bar{Z}\, e^{-i\alpha}) = e^{-in\alpha}\Psi(Z, \bar{Z})\}$$

is orthogonal to that for $n = 0$ (absence of interaction) (see [13])

$$\langle \Psi_n, \Psi_0 \rangle = \int_P \Psi_0^*(Z, \bar{Z})\Psi_n(Z, \bar{Z})\, dV(Z, \bar{Z}) \qquad \Psi_n, \Psi_0 \in \bigoplus_{p \in \mathbb{Z}} \Im_p$$

$$\Psi_n \in \Im_n \qquad \Psi_0 \in \Im_0.$$

Remark. The equivariance condition $\Psi(Z\, e^{i\alpha}, \bar{Z}\, e^{-i\alpha}) = e^{-in\alpha}\, \Psi(Z, \bar{Z})$ implies in particular that the functions of class $\Im_n$ are defined on a bundle space which is contracted to the 3D lens space L_n^3. This L_n^3 is quite different from $\mathbb{R}^3 \backslash \{0\}$ which is the domain of $\Im_0$. For instance $\pi_1(L_n^3) = \mathbb{Z}_n \neq \pi_1(\mathbb{R}^3 \backslash \{0\}) = \pi_1(S^2) = 0$. So the domains of $\Im_n$ and $\Im_0$ are not simply non-equivalent bundles, they are different as topological spaces!

Thus, the forgetting of the interaction between the monopole of one dyon and the charge of another, is a transition to a quite different space of functions. That is why the behaviour of asymptotic wavefunctions under a dyon permutation on the bundles considered in [12] cannot be taken as a basis for a conclusion about dyon statistics.

The example of space reflection shows how radically this transition can change the properties with respect to a discrete transformation. There is no parity transformation on $\Im_n$, but it does exist on $\Im_0$. So it is impossible to remove the interaction by some continuous procedure. In contrast to the usual quantum mechanics (on trivial bundles), this problem must be analysed directly for interacting dyons.

Acknowledgments

The authors would like to thank Professor I S Shapiro and M A Solov'ev for helpful discussions.

Appendix. Proof of non-simple connectedness

Since dyons are regarded as being far from each other (at infinity) and their 'internal structure' is fixed, the configurations are determined by the position of the two dyons on the sphere S^2 at infinity. A path in the configuration space is a motion of the two dyons on S^2, hence it is represented by a pair of paths on S^2. Let us suppose that there exists a homotopy

$$h:[0,2\pi]\times[0,1]\rightarrow S^2\times S^2\times[0,2\pi]$$

which transforms the 2π clockwise rotation of dyons (represented by the *pair* of paths shown on figure 3(a)) to the identity

$$h:(s,0)\rightarrow(2\pi\,\mathrm{rot}(s),s)$$

$$h:(s,1)\rightarrow(\mathrm{id}(s),s).$$

Let us consider the configuration $C\in S^2\times S^2$, in which one of the dyons is located exactly under the other one. This is the case when one of the vectors $\boldsymbol{r}_{12}$ and $\boldsymbol{r}_{21}$ has coordinates $(0,0,z<0)$ and, consequently, this configuration does not belong to the chart U_{++++} (by the definition of U_{++++}), hence it does not belong to its subset $U_{++++}{}^{\mathrm{as}}$. Now we will show that the deformation of the 2π rotation at the homotopy h contains this configuration, i.e. there exist t_0 and s_0 such that

$$h(s_0,t_0)\rightarrow(C,s_0).$$

With this aim let us consider these pairs of paths as curves in the cross-product $S^2\times[0,2\pi]$. Since it does not matter which dyon is under the other, it suffices to consider $D^2\times[0,2\pi]$ where (D^2 is obtained from S^2 by identifying the points (x,y,z) and $(x,y,-z)\in S^2$: $D^2=\{(x,y)\,|\,(x,y,z)\in S^2\}$). A path in configuration space B^{as} is represented by pair of curves with initial points on $D^2\times\{0\}$ and final points on $D^2\times\{2\pi\}$ (figure 3(b)). Under the homotopy h the linked curves 1_γ and 2_γ must intersect to become unlinked as 1_{id} and 2_{id}. The intersection point is C in the configuration space.

References

[1] Wu T T and Yang C N 1975 *Phys. Rev.* D **12** 3845
[2] Greub W and Petry H R 1975 *J. Math. Phys.* **216** 1347
[3] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol 1 (New York: Academic)

[4] Solov'ev M A 1984 *JETP Lett.* **39** 714
[5] Goldhaber A S 1976 *Phys. Rev. Lett.* **36** 1122
[6] Horvathy P A 1981 *Int. J. Theor. Phys.* **20** 697
[7] Coleman S 1981 The magnetic monopole fifty years later *Preprint* Harvard University, Cambridge
[8] Feynman R P 1987 *Dirac Memorial Lectures* Cambridge University Press, pp 1-59
[9] Souriau J-M 1970 *Structure des Systèmes Dynamiques* (Paris: Dunod)
[10] Savinkov A G and Shapiro I S 1988 *JETP Lett.* **47** 1116
[11] Friedman J L and Sorkin R D 1979 *Phys. Rev.* D **20** 2511
[12] Friedman J L and Sorkin R D 1980 *Commun. Math. Phys.* **73** 161
[13] Savinkov A G with Ryzhov A B 1991 *Int. J. Mod. Phys.* A **6** 577